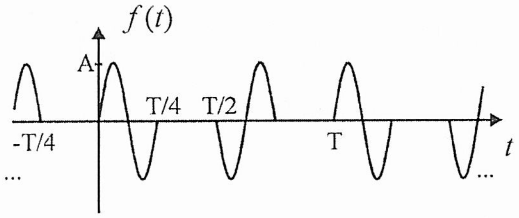
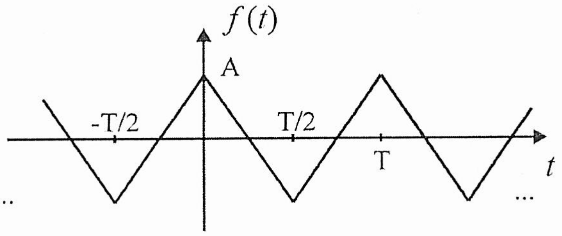
Kolokwium nr 1 z Teorii Sygnałów

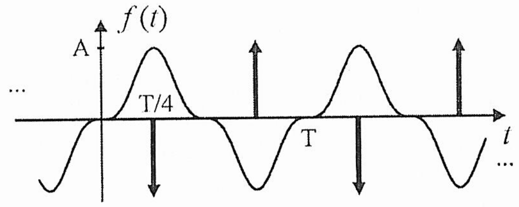
1. Proszę wyznaczyć współczynniki F_k rozwinięcia w zespolony szereg Fouriera dla sygnału okresowego pokazanego na rysunku. Sygnał powstał przez przekształcenie sinusoidy. Proszę **narysować widmo amplitudowe i fazowe** tego sygnału.



2. Proszę wyznaczyć współczynniki F_k rozwinięcia w zespolony szereg Fouriera dla sygnału okresowego trójkątnego pokazanego na rysunku.



3. Proszę wyznaczyć współczynniki F_k rozwinięcia w zespolony szereg Fouriera dla pokazanego na rysunku sygnału okresowego, który jest **sumą** funkcji $A\sin^3(\omega_0 t)$ oraz ciągu naprzemiennych impulsów Diraca. Dla jakiej wartości A sygnał nie posiada pierwszej harmonicznej?



4. Proszę obliczyć wartość średnią sygnału $f(t) = A\sin^6(t)$, wiedząc, że jego wartość skuteczna wynosi $U = \frac{\sqrt{2}}{2}$. Uwaga: rozwiązanie jest liczbą.

Zadanie 1

$$f(t)=\begin{cases} A\sin\frac{2\pi\cdot 4}{T}\cdot t & \text{dla } t\in(0;\frac{T}{4}\rangle \\ 0 & \text{dla } t\in(\frac{T}{4};\frac{T}{2}\rangle \\ -A\sin\frac{2\pi\cdot 4}{T}(t-\frac{T}{2}) & \text{dla } t\in(\frac{T}{2};\frac{3}{4}T\rangle \\ 0 & \text{dla } t\in(\frac{3}{4}T;T\rangle \end{cases}$$

sygnał jest antysymetryczny

$$F_K = F_K'\left(1-(-1)^k\right)$$

$$f'(t)=\begin{cases} A\sin\left(\frac{8\pi}{T}t\right) & \text{dla } t\in(0;\frac{T}{4}\rangle \\ 0 & \text{dla } t\in(\frac{T}{4};T\rangle \end{cases}$$

$$F_k' = \frac{1}{T}\int_0^{\frac{T}{4}} A\sin\left(\frac{8\pi}{T}t\right)e^{-j\frac{2\pi}{T}kt}dt = \frac{A}{T}\int_0^{\frac{T}{4}}\frac{e^{j\frac{8\pi}{T}t}-e^{-j\frac{8\pi}{T}t}}{2j}e^{-j\frac{2\pi}{T}kt}dt =$$

$$=\frac{A}{2Tj}\int_0^{\frac{T}{4}}\left[e^{j\frac{2\pi}{T}(4-k)t}-e^{-j\frac{2\pi}{T}(4+k)t}\right]dt =$$

$$=\frac{A}{2Tj}\left[\frac{e^{j\frac{2\pi}{T}(4-k)t}}{j\frac{2\pi}{T}(4-k)}\Bigg|_0^{\frac{T}{4}} - \frac{e^{-j\frac{2\pi}{T}(4+k)t}}{-j\frac{2\pi}{T}(4+k)}\Bigg|_0^{\frac{T}{4}}\right] =$$

$$=\frac{A}{2\cancel{T}j}\left[\frac{\cancel{T}\left(e^{j\frac{2\pi}{\cancel{T}}(4-k)\cdot\frac{\cancel{T}}{4}}-1\right)}{j2\pi(4-k)} + \frac{\cancel{T}\left(e^{-j\frac{2\pi}{T}(4+k)\frac{T}{4}}-1\right)}{j2\pi(4+k)}\right] =$$

$$=-\frac{A}{4\pi}\left[\frac{e^{j\frac{\pi}{2}(4-k)}-1}{(4-k)} + \frac{e^{-j\frac{\pi}{2}(4+k)}-1}{4+k}\right]$$

$$F_K = -\frac{A}{4\pi}\left[\frac{e^{j\frac{\pi}{2}(4-k)}-1}{4-k} + \frac{e^{-j\frac{\pi}{2}(4+k)}-1}{4+k}\right]\left(1-(-1)^K\right) \qquad K\neq -4,\ K\neq 4$$

$$F_k = -\frac{A}{4\pi}\left[\frac{e^{j\frac{\pi}{2}\cdot 4}\cdot e^{-j\frac{\pi}{2}k} - 1}{4-k} + \frac{e^{-j\frac{\pi}{2}\cdot 4}e^{-j\frac{\pi}{2}k} - 1}{4+k}\right]\left(1-(-1)^k\right) =$$

$$= -\frac{A}{4\pi}\left[\frac{e^{-j\frac{\pi}{2}k}-1}{4-k} + \frac{e^{-j\frac{\pi}{2}k}-1}{4+k}\right]\left(1-(-1)^k\right) =$$

$$= -\frac{A}{4\pi}\left[\frac{4\cdot e^{-j\frac{\pi}{2}k} \cancel{+k e^{-j\frac{\pi}{2}k}} - 4 \cancel{-k} + 4\cdot e^{-j\frac{\pi}{2}k} \cancel{-k e^{-j\frac{\pi}{2}k}} - 4 \cancel{+k}}{16-k^2}\right]\left(1-(-1)^k\right) =$$

$$= -\frac{A}{\cancel{4}\pi}\left[\frac{\cancel{8}^{2} e^{-j\frac{\pi}{2}k} - \cancel{8}^{2}}{16-k^2}\right]\left(1-(-1)^k\right) = -\frac{A}{\pi}\left(\frac{2e^{-j\frac{\pi}{2}k}-2}{16-k^2}\right)\left(1-(-1)^k\right) \qquad \begin{matrix} k \neq -4 \\ k \neq 4 \end{matrix}$$

$F_0 = 0$ sygnał antysymetryczny

$$F_4' = \frac{1}{T}\int_0^{\frac{T}{4}} A\sin\left(\frac{8\pi}{T}t\right)e^{-j\frac{2\pi}{T}\cdot 4t}dt = \frac{A}{T}\int_0^{\frac{T}{4}}\frac{e^{j\frac{8\pi}{T}t}-e^{-j\frac{8\pi}{T}t}}{2j}e^{-j\frac{8\pi}{T}t}dt =$$

$$= \frac{A}{T}\int_0^{\frac{T}{4}}\frac{1-e^{-j16\frac{\pi}{T}t}}{2j}dt = \frac{A}{2Tj}\left[\int_0^{\frac{T}{4}}dt - \int_0^{\frac{T}{4}}e^{-j16\frac{\pi}{T}t}dt\right] =$$

$$= \frac{A}{2Tj}\left[\frac{T}{4} + \frac{e^{-j16\frac{\pi}{T}t}}{j16\frac{\pi}{T}}\Bigg|_0^{\frac{T}{4}}\right] = \frac{A}{2\cancel{T}j}\left[\frac{\cancel{T}}{4} + \frac{\cancel{T}}{16\pi j}\left(e^{-j\cancel{16}^{4}\frac{\pi}{\cancel{T}}\cdot\frac{\cancel{T}}{\cancel{4}}} - 1\right)\right] =$$

$$= \frac{A}{8j} + \frac{A}{32\pi j}\left(e^{-j4\pi}-1\right) = \frac{A}{8j} + \frac{A}{32\pi j}(1-1) = \frac{A}{8j}$$

$$F_{-4}' = \frac{1}{T}\int_0^{\frac{T}{4}} A\sin\left(\frac{8\pi}{T}t\right)e^{-j\frac{2\pi}{T}\cdot(-4)t}dt = \frac{A}{T}\int_0^{\frac{T}{4}}\frac{e^{j\frac{8\pi}{T}t}-e^{-j\frac{8\pi}{T}t}}{2j}e^{j\frac{8\pi}{T}t}dt =$$

$$= \frac{A}{T}\int_0^{\frac{T}{4}}\frac{e^{j\frac{16\pi}{T}t}-1}{2j}dt = \frac{A}{2Tj}\left[\frac{e^{j\frac{16\pi}{T}t}}{j16\pi}-t\right]_0^{\frac{T}{4}} = \frac{A}{2\cancel{T}j}\left[\frac{\cancel{T}e^{j\frac{16\pi}{\cancel{T}}\cdot\frac{\cancel{T}}{4}}}{j16\pi} - \frac{\cancel{T}}{4} - \frac{\cancel{T}}{j16\pi}\right] =$$

$$= \cancel{\frac{A}{-32\pi}} - \frac{A}{8j} \cancel{+\frac{A}{32\pi}} = -\frac{A}{8j}$$

$$F_4 = F_4'(1-(-1)^k) = \frac{A}{8j}\left(1-(-1)^k\right)$$

$$F_{-4} = F_{-4}'(1-(-1)^k) = -\frac{A}{8j}\left(1-(-1)^k\right)$$

$$F_k = \begin{cases} 0 & \text{dla} \quad k=0 \\ \frac{A}{8j}\left(1-(-1)^k\right) & \text{dla } k=4 \\ -\frac{A}{8j}\left(1-(-1)^k\right) & \text{dla } k=-4 \\ -\frac{A}{\pi}\left(\frac{2e^{-j\frac{\pi}{2}k}-2}{16-k^2}\right)\left(1-(-1)^k\right) & \text{dla} \quad \begin{matrix} k\neq -4 \\ k\neq 4 \\ k\neq 0 \end{matrix} \end{cases}$$

$F_0 = 0$ $\qquad |F_0| = 0 \qquad \varphi_0 = 0$

$F_1 = -\frac{A}{\pi}\left(\frac{2e^{-j\frac{\pi}{2}}-2}{16-1^2}\right)(1+1) = -\frac{A}{\pi}\left(\frac{-2j-2}{15}\right)\cdot 2 = \frac{4A}{15\pi}(j+1)$ $\qquad |F_1| = \frac{4A}{15\pi}\sqrt{2} \qquad \varphi_1 = \operatorname{arctg}\left(\frac{1}{1}\right) = 0.785 = \frac{\pi}{4}$

$F_{-1} = -\frac{A}{\pi}\left(\frac{2e^{+j\frac{\pi}{2}}-2}{16-1^2}\right)(1+1) = -\frac{A}{\pi}\left(\frac{2j-2}{15}\right)\cdot 2 = \frac{4A}{15\pi}(-j+1)$ $\qquad |F_{-1}| = \frac{4A}{15\pi}\sqrt{2} \qquad \varphi_{-1} = \operatorname{arctg}\left(\frac{-1}{1}\right) = -0.785 = -\frac{\pi}{4}$

$F_2 = -\frac{A}{\pi}\left(\frac{2e^{-j\frac{\pi}{2}\cdot 2}-2}{16-2^2}\right)(1-1) = 0$ $\qquad |F_2| = 0$

$|F_{-2}| = 0$

$F_{-2} = 0$

$F_3 = -\frac{A}{\pi}\left(\frac{2e^{-j\frac{\pi}{2}3}-2}{16-3^2}\right)(1+1) = -\frac{2A}{\pi}\left(\frac{j-1}{7}\right)\cdot 2 = \frac{4A}{7\pi}(1-j)$ $\qquad |F_3| = \frac{4A}{7\pi}\sqrt{2} \qquad \varphi_3 = \operatorname{arctg}\left(\frac{1}{-1}\right) = -0.785 = -\frac{\pi}{4}$

$F_{-3} = -\frac{A}{\pi}\left(\frac{2e^{j\frac{\pi}{2}3}-2}{16-3^2}\right)(1+1) = -\frac{2A}{\pi}\left(\frac{-j-1}{7}\right)\cdot 2 = \frac{4A}{7\pi}(1+j)$ $\qquad |F_{-3}| = \frac{4A}{7\pi}\sqrt{2} \qquad \varphi_{-3} = \operatorname{arctg}\left(\frac{1}{1}\right) = 0.785 = \frac{\pi}{4}$

$F_4 = \frac{A}{8j}$ $\qquad |F_4| = \frac{A}{8} \qquad \varphi_4 = \frac{\pi}{2}$

$F_{-4} = -\frac{A}{8j}$ $\qquad |F_{-4}| = \frac{A}{8} \qquad \varphi_{-4} = -\frac{\pi}{2}$

$F_5 = -\frac{A}{\pi}\left(\frac{2e^{-j\frac{\pi}{2}\cdot 5}-2}{16-5^2}\right)(1+1) = -\frac{2A}{\pi}\left(\frac{-j-1}{-9}\right)\cdot 2 = \frac{4A}{9\pi}(-j-1)$ $\qquad |F_5| = \frac{4A}{9\pi}\sqrt{2} \qquad \varphi_5 = \frac{\pi}{4}$

$F_{-5} = -\frac{A}{\pi}\left(\frac{2e^{j\frac{\pi}{2}5}-2}{16-5^2}\right)(1+1) = -\frac{2A}{\pi}\left(\frac{j-1}{-9}\right)\cdot 2 = \frac{4A}{9\pi}(j-1)$ $\qquad |F_{-5}| = \frac{4A}{9\pi}\sqrt{2} \qquad \varphi_{-5} = -\frac{\pi}{4}$

Zadanie 2

$$f(t) = \begin{cases} -\frac{4A}{T}t + A & \text{dla } t \in (0; \frac{T}{2}) \\ -\left[-\frac{4A}{T}(t - \frac{T}{2}) + A\right] & \text{dla } t \in (\frac{T}{2}; T) \end{cases}$$

sygnał antysymetryczny

$$f'(t) = \begin{cases} -\frac{4A}{T}t + A & \text{dla } t \in (0; T/2) \\ 0 & \text{dla } t \in (\frac{T}{2}; T) \end{cases}$$

$$F_k = F_k'(1 - (-1)^k)$$

$$F_k' = \frac{1}{T}\int_0^T f(t) e^{-j\frac{2\pi}{T}\cdot k\cdot t} dt = \frac{1}{T}\int_0^{\frac{T}{2}} \left(-\frac{4A}{T}\cdot t + A\right) e^{-j\frac{2\pi}{T}\cdot k\cdot t} dt =$$

$$= -\frac{4A}{T^2}\int_0^{\frac{T}{2}} t e^{-j\frac{2\pi}{T}kt} dt + \frac{A}{T}\int_0^{\frac{T}{2}} e^{-j\frac{2\pi}{T}kt} dt =$$

$$= -\frac{4A}{T^2}\left(\frac{t e^{-j\frac{2\pi}{T}kt}}{-j\frac{2\pi}{T}k} + \frac{e^{-j\frac{2\pi}{T}kt}}{\frac{4\pi^2}{T^2}k^2}\right)\Bigg|_0^{\frac{T}{2}} + \frac{A}{T}\frac{e^{-j\frac{2\pi}{T}kt}}{-j\frac{2\pi}{T}\cdot k}\Bigg|_0^{\frac{T}{2}} =$$

$$= -\frac{4A}{T^2}\left(\frac{\frac{T}{2}e^{-j\frac{2\pi}{T}k\frac{T}{2}}}{-j\frac{2\pi}{T}k} + \frac{T^2 e^{-j\frac{2\pi}{T}k\frac{T}{2}}}{4\pi^2k^2} - \frac{T^2}{4\pi^2k^2}\right) + \frac{A}{T}\frac{T}{-j2\pi k}\left(e^{-j\frac{2\pi}{T}k\frac{T}{2}} - 1\right) =$$

$$= -\frac{\cancel{4}A}{\cancel{T^2}}\left(\frac{\cancel{T^2}e^{-j\pi k}}{-\cancel{4}j\pi k} + \frac{\cancel{T^2}e^{-j\pi k}}{\cancel{4}\pi^2k^2} - \frac{\cancel{T^2}}{\cancel{4}\pi^2k^2}\right) + \frac{A}{-j2\pi k}\left(e^{-j\pi k} - 1\right) =$$

$$= -A\left(\cancel{\frac{e^{-j\pi k}}{-j\pi k}} + \frac{e^{-j\pi k}}{\pi^2k^2} - \frac{1}{\pi^2k^2}\right) + \cancel{\frac{A}{-j2\pi k}e^{-j\pi k}} - \frac{A}{-j2\pi k} =$$

$$= +\frac{Ae^{-j\pi k}}{j\pi k} - \frac{\frac{1}{2}Ae^{-j\pi k}}{j\pi k} - \frac{Ae^{-j\pi k}}{\pi^2k^2} + \frac{A}{\pi^2k^2} + \frac{\frac{1}{2}A}{j\pi k} =$$

$$= \frac{\frac{1}{2}Ae^{-j\pi k}}{j\pi k} + \frac{\frac{1}{2}A}{j\pi k} - \left(\frac{Ae^{-j\pi k} + A}{\pi^2k^2}\right) = A\left(e^{-j\pi k} + 1\right)\left(\frac{1}{2\pi jk} - \frac{1}{\pi^2k^2}\right) = A\left(1 + (-1)^k\right)\left(\frac{1}{2\pi jk} - \frac{1}{\pi^2k^2}\right)$$

$$F_k = A\left(1+(-1)^k\right)\left(\frac{1}{2\pi jk} - \frac{1}{\pi^2 k^2}\right)\left(1-(-1)^k\right) =$$

$$= A\left(1-(-1)^{2k}\right)\left(\frac{1}{2\pi jk} - \frac{1}{\pi^2 k^2}\right)$$

$F_0 = 0$ sygnał antysymetryczny.

Zadanie 3

$$f(t) = A\sin^3(\omega_0 t) + A\delta(t - \tfrac{3}{4}T) - A\delta(t - \tfrac{T}{4}) = f'(t) + f''(t) + f'''(t)$$

$$\begin{array}{ccccccc} & & & 1 & & & \\ & & 1 & & 1 & & \\ & 1 & & 2 & & 1 & \\ 1 & & 3 & & 3 & & 1 \end{array}$$

$$f'(t) = A\sin^3(\omega_0 t) = A\left(\frac{e^{j\omega_0 t} - e^{-j\omega_0 t}}{2j}\right)^3 = -\frac{A}{8j}\left(e^{3j\omega_0 t} - 3e^{j\omega_0 t} + 3e^{-j\omega_0 t} - e^{-3j\omega_0 t}\right)$$

$$F_3 = -\frac{A}{8j} \qquad F_{-3} = \frac{A}{8j}$$

$$F_1 = \frac{3A}{8j} \qquad F_{-1} = -\frac{3A}{8j}$$

$$f''(t) = A\delta(t - \tfrac{3}{4}T) - A\delta(t - \tfrac{T}{4})$$

$$F_k'' = \frac{1}{T}\int_0^T A\delta(t - \tfrac{3}{4}T)e^{-j\frac{2\pi}{T}kt}dt = \frac{A}{T}e^{-j\frac{2\pi}{T}k\cdot\frac{3T}{4}} = \frac{A}{T}e^{-j\frac{3}{2}\pi k}$$

$$F_k''' = \frac{1}{T}\int_0^T -A\delta(t - \tfrac{T}{4})e^{-j\frac{2\pi}{T}kt}dt = -\frac{A}{T}e^{-j\frac{2\pi}{T}k\cdot\frac{T}{4}} = -\frac{A}{T}e^{-j\frac{\pi}{2}k}$$

$$\Rightarrow \frac{2A}{T}\left(\frac{e^{-j\frac{3}{2}\pi k} - e^{-j\frac{\pi}{2}k}}{2}\right)$$

$$F_k = \begin{cases} F_0 = 0 \\ F_1 = \frac{3A}{8j} + \frac{A}{T}e^{-j\frac{3}{2}\pi k} - \frac{A}{T}e^{-j\frac{\pi}{2}k} \\ F_{-1} = -\frac{3A}{8j} + \frac{A}{T}e^{-j\frac{3}{2}\pi k} - \frac{A}{T}e^{-j\frac{\pi}{2}k} \\ F_3 = -\frac{A}{8j} + \frac{A}{T}e^{-j\frac{3}{2}\pi k} - \frac{A}{T}e^{-j\frac{\pi}{2}k} \\ F_3 = \frac{A}{8j} + \frac{A}{T}e^{-j\frac{3}{2}\pi k} - \frac{A}{T}e^{-j\frac{\pi}{2}k} \\ F_k = \frac{A}{T}e^{-j\frac{3}{2}\pi k} - \frac{A}{T}e^{-j\frac{\pi}{2}k} \qquad k \neq 1, -1, 3, -3, 0 \end{cases}$$

$$F_1 = \frac{3A}{8j} + \frac{A}{T}e^{-j\frac{3}{2}\pi} - \frac{A}{T}e^{-j\frac{\pi}{2}} = \frac{3A}{8j} + \frac{A}{T}j + \frac{A}{T}j = \frac{3A}{8j} + \frac{2A}{T}j$$

Zadanie 4

$$f(t) = A\sin^6(t) = A\left(\frac{e^{jt}-e^{-jt}}{2j}\right)^6 =$$

$$= \frac{-A}{2^6}\left[e^{6jt} - 6e^{4jt} + 15e^{2jt} - 20 + 15e^{-2jt} - 6e^{-4jt} + e^{-6jt}\right] =$$

$$= -\frac{A}{2^6}e^{6jt} + \frac{6A}{2^6}e^{4jt} - \frac{15A}{2^6}e^{2jt} + \underbrace{\frac{20A}{2^6}}_{F_0} - \frac{15A}{2^6}e^{-2jt} + \frac{6A}{2^6}e^{-4jt} - \frac{A}{2^6}e^{-6jt}$$

1
1 1
1 2 1 $n=2$
1 3 3 1 $n=3$
1 4 6 4 1 $n=4$
1 5 10 10 5 1 $n=5$
1 6 15 20 15 6 1 $n=6$

$F_0 = \frac{20A}{2^6}$

$F_2 = -\frac{15A}{2^6}$ $\quad F_{-2} = \frac{15A}{2^6}$

$F_4 = \frac{6A}{2^6}$ $\quad F_{-4} = \frac{6A}{2^6}$

$F_6 = -\frac{A}{2^6}$ $\quad F_{-6} = \frac{A}{2^6}$

$$P = \sum_{k=-\infty}^{\infty} |F_k|^2 =$$

$$= \left(\frac{20}{2^6}A\right)^2 + 2\left(\frac{15A}{2^6}\right)^2 + 2\left(\frac{6A}{2^6}\right)^2 + 2\left(\frac{A}{2^6}\right)^2 =$$

$$= \left(\frac{200}{2^{12}} + 2\cdot\frac{225}{2^{12}} + 2\frac{36}{2^{12}} + 2\right)A^2 =$$

$$= \frac{562}{2^{11}}A^2 = \frac{231}{2^{10}}A^2$$

1 1
200
225
36
281
562 :2
4
16

$$U^2 = P$$

$$\frac{2}{4} = \frac{231}{2^{10}}A^2$$

$$\frac{\cancel{2^{11}}\,2^9}{\cancel{4}\cdot 231} = A^2 \quad \Rightarrow A = \sqrt{\frac{2^9}{231}} = 2^4\sqrt{\frac{2}{231}}$$

$$F_0 = \frac{\cancel{20}^{5}}{\cancel{2^6}_{1}}\cdot\frac{2^4\cdot\sqrt{2}}{\sqrt{231}} = \underline{\frac{5\cdot\sqrt{2}}{\sqrt{231}}}$$